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Potentials With Zero Scattering Phase

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Abstract

The Gelfand-Levitan equation is used to construct potentials which, for a given angular momentum, yield scattering phases which are identically zero for all energies.

The technique used herein gives all such potentials which belong to the class of potentials for which f(k) is analytic in the lower half-plane. In these cases the potentials support point eigenvalues which are either zero or are positive and embedded in the continuous spectrum. In the case of $\ell = 0$, the potential which supports the zero point eigenvalue was given by Bargmann who used a different technique.

1. The Gelfand-Levitan equation for zero angular momentum

The Gelfand-Levitan equation was introduced in Reference [1] as a device for obtaining scattering potentials from the spectral measure function. In Reference [2] it was shown how the spectral measure function may be obtained from the scattering phase and hence how one may obtain the scattering potential from the scattering phases and the point eigenvalues and normalizations of the eigenfunctions belonging to the point eigenvalues.

Explicitly, the Gelfand-Levitan equation for $\ell = 0$ is

$$(1.1)\begin{cases} K(r,r') = -D(r,r') - \displaystyle\int_0^r K(r,r'')\, D(r'',r')\, dr'' \\ \text{where} \\ D(r,r') = \dfrac{2}{\pi}\displaystyle\int_0^\infty \left[|f(k)|^{-2} - 1 \right] \sin kr \sin kr'\, dk \\ \qquad + \displaystyle\sum_{i=1}^{n} C_i^{-1}\, g(r,E_i) \cdot g^*(r',E_i). \end{cases}$$

The values E_i correspond to prescribed point eigenvalues. The functions $g(r,E_i)$ are given by

$$g(r,E_i) = A_i e^{i\pi/4} \frac{\sinh \sqrt{-E_i}\, r}{\sqrt{\pi}\,(-E_i)^{1/4}} \quad , \quad (E_i < 0)$$

$$(1.1a) \qquad = \frac{A_i \sin \sqrt{E_i}\, r}{\sqrt{\pi}\,(E_i)^{1/4}} \quad , \quad (E_i > 0)$$

$$= \frac{A_i}{\sqrt{\pi}}\, r \quad , \quad (E_i = 0) \quad .$$

The constants C_i are prescribed normalizations for the eigenstates corresponding to the point eigenvalues, while the constants A_i are arbitrary complex numbers which one must also prescribe.

Gelfand and Levitan show that if one prescribes the function $|f(k)|$, the point eigenvalues E_i and the normalization constants C_i, and if one solves (1.1) for $K(r,r')$, then on writing

$$
\begin{aligned}
V(r) &= 2\frac{d}{dr} K(r,r) \quad , \\
\Psi(r,k) &= \sqrt{\frac{2}{\pi}} \sin kr + \sqrt{\frac{2}{\pi}} \int_0^r K(r,r') \sin kr' dr' \quad , \qquad (1.2) \\
\Psi(r,E_i) &= g(r,E_i) + \int_0^r K(r,r')\, g(r',E_i) dr' \quad .
\end{aligned}
$$

The real functions $\Psi(r,k)$ are eigenfunctions of the operator $-\frac{d^2}{dr^2} + V(r)$ belonging to the continuous spectrum corresponding to the eigenvalue k^2, whereas the generally complex functions $\Psi(r,E_i)$ are eigenfunctions corresponding to the point eigenvalue E_i.

Furthermore, the eigenfunctions satisfy the orthogonality conditions

$$
\begin{aligned}
&\int_0^\infty \Psi(r,k)\, \Psi(r,k')\, dr = |f(k)|^2\, \delta(k - k') \\
(1.3) \quad &\int_0^\infty \Psi^*(r,E_i)\, \Psi(r,E_j)\, dr = C_i\, \delta_{ij} \\
&\int_0^\infty \Psi(r,E_i)\, \Psi(r,k)\, dr = 0 \quad .
\end{aligned}
$$

and the completeness relations

$$
\begin{aligned}
&\int_0^\infty \Psi(r,k)\, |f(k)|^{-2}\, \Psi(r',k) dk + \sum_{i=1}^{n} C_i^{-1}\, \Psi(r,E_i)\, \Psi^*(r',E_i) \\
(1.4) \quad &= \frac{2}{\pi} \int_0^\infty \sin kr \sin kr'\, dk = \delta(r - r') \quad .
\end{aligned}
$$

The function f(k) is related to the phase δ in the following way for $k > 0$:

$$(1.5) \qquad e^{2i\delta(k)} = \frac{f(k)}{f^*(k)}$$

or

$$(1.5a) \qquad f(k) = |f(k)| \, e^{i\delta(k)} .$$

It can also be shown that

$$(1.6) \qquad \lim_{r \to \infty} \Psi(r,k) = \sqrt{\frac{2}{\pi}} \, |f(k)| \sin (kr + \delta)$$

Hence, on using (1.6) and (1.5a), one could obtain f(k).

In Reference [2], in order that one may obtain f(k) uniquely from the phase by means of (1.5), the following assumptions are made:

1. $f(-k) = f(k)^*$ for k real.
2. f(k) is analytic in the lower half plane.
3. $\lim_{|k| \to \infty} f(k) = 1$, k in the lower half plane.
4. The point eigenvalues of the operator $-\frac{d^2}{dr^2} + V(r)$ are given by the zeros of f(k) on the negative imaginary axis.

A class of potentials which lead to such functions f(k) are discussed in Reference [3].

It should be pointed out that these conditions are not necessary to obtain the potential V(r) from the spectral measure function or, equivalently, from $|f(k)|$, C_i and E_i. In Reference [4] Kay and Moses have shown that it is only necessary for the Gelfand-Levitan equation to have a solution in order to reproduce the prescribed spectral measure function. It is only for the purpose of obtaining spectral measure function from the phase δ that one must impose some extra conditions.

For the cases in which the phase δ is identically zero for all k, the conditions above are too severe to admit any potential but the trivial one $V \equiv 0$. Therefore, to get non-trivial results we shall relax the conditions on f(k). Specifically we shall drop condition 4 which relates the zeros of f(k) to the point eigenvalues. In doing so we are following a technique which was used by Kay and Moses [5] to obtain reflectionless potentials in the one-dimensional case $(-\infty < x < \infty)$. We shall be led to potentials which, while in a certain sense pathological, are perfectly smooth and die sufficiently rapidly at infinity to permit scattering to be defined. Thus we shall consider the class of potentials which lead to functions f(k) which satisfy the first three requirements and find all the potentials of this class which lead to scattering with zero phase.

We shall now proceed to derive the phaseless potentials.

From (1.5) we see that if $\delta(k)$ is identically zero, for real $k \geq 0$, f(k) is real on the positive real axis. On using conditions 1, 2, and 3 above and the reflection theorem of analysis, we come to the conclusion that a necessary and sufficient condition on f(k) is that

$$(1.7) \qquad f(k) \equiv 1 \text{ on the whole complex k plane.}$$

If we accepted condition 4 on f(k), namely that bound states correspond to zeros of f(k) on the negative imaginary axis, we see that, since in the present case there are no zeros, there should be no bound states. Under these conditions we would have $D(r,r') \equiv 0$ and hence also $K(r,r') \equiv 0$ and $V(r) \equiv 0$. We shall therefore assume the possibility of bound states in order to obtain non-trivial potentials. We note, however, that in (1.1) only the absolute value of f(k) appears. If we admit bound states, we shall expect to obtain potentials which reproduce the given spectral measure function and find in particular that $|f(k)|=1$. In general, however, the phase will not be zero identically. Hence, it will be our objective to find those bound states or combinations of bound states which will lead to zero phase.

Let us first consider the case in which a single bound state exists with a corresponding point eigenvalue E_1. In such cases (1.1) can be solved by separation of variables, i.e., by setting $K(r,r') = H(r) \sinh \sqrt{-E_1}\, r'$ and solving for the functions $H(r)$.

For $E_1<0$, we have

$$D(r,r') = C_1^{-1} |A_1|^2 \frac{\sinh \sqrt{-E_1}\, r \sinh \sqrt{-E_1}\, r'}{\pi \sqrt{-E_1}} \tag{1.8}$$

$$K(r,r') = \frac{-\sinh \sqrt{-E_1}\, r \sinh \sqrt{-E_1}\, r'}{\left[\mu + \frac{1}{2}\left(\frac{\sinh 2\sqrt{-E_1}\, r}{2\sqrt{-E_1}} - r\right)\right]} , \tag{1.9}$$

where

$$\mu = \frac{\pi C_1 \sqrt{-E_1}}{|A_1|^2} . \tag{1.9a}$$

$$V(r) = 2\frac{d}{dr} K(r,r) = \frac{2 \sinh \sqrt{-E_1}\, r \left[2\sqrt{-E_1}\left(\frac{r}{2} - \mu\right) \cosh \sqrt{-E_1}\, r - \sinh \sqrt{-E_1}\, r\right]}{\left[\mu + \frac{1}{2}\left(\frac{\sinh 2\sqrt{-E_1}\, r}{2\sqrt{-E_1}} - r\right)\right]^2} \tag{1.10}$$

$$\Psi(r,k) = \sqrt{\frac{2}{\pi}} \sin kr - \sqrt{\frac{2}{\pi}} \frac{\sinh \sqrt{-E_1}\, r \left(\sqrt{-E_1} \sin kr \cosh kr - k \cos kr \sinh \sqrt{-E_1}\, r\right)}{\left[\mu + \frac{1}{2}\left(\frac{\sinh 2\sqrt{-E_1}\, r}{2\sqrt{-E_1}} - r\right)\right] (k^2 - E_1)} \tag{1.11}$$

$$(1.12) \quad \Psi(r,E_1) = \frac{A_1 e^{i\pi/4} \sinh \sqrt{-E_1}\, r}{\sqrt{\pi}\,(-E_1)^{1/4}} \qquad \frac{\mu}{\left[\mu + \frac{1}{2}\left(\frac{\sinh 2\sqrt{-E_1}\, r}{2\sqrt{-E_1}} - r\right)\right]} \; .$$

On inspecting the results, it is clear that the potential decays exponentially for large r. Furthermore, it can be shown by direct substitution that the eigenfunctions (1.11) and (1.12) really are eigenfunctions of the operator $- \frac{d^2}{dr^2} + V(r)$.

However, we see that the scattering phase is not zero, for

$$(1.13) \quad \lim_{r \to \infty} \Psi(r,k) = \sqrt{\frac{2}{\pi}} \left[\sin kr + \frac{2}{(k^2 - E_1)} \quad (E_1 \sin kr + k\sqrt{-E_1} \cos kr) \right]$$

$$= \sqrt{\frac{2}{\pi}} \quad \sin\,(kr + \delta) \quad ,$$

where

$$(1.14) \qquad \cos\delta = \frac{(k^2 + E_1)}{(k^2 - E_1)} \quad , \quad \sin\delta = \frac{2k\sqrt{-E_1}}{(k^2 - E_1)} \quad .$$

In the present case the absolute value of f(k) is 1 on the real k axis as we would expect from the Gelfand-Levitan equation. However f(k) itself is not 1. In fact on the real axis

$$(1.15) \qquad f(k) = e^{i\delta} = \cos\delta + i \sin\delta = \frac{(k + i\sqrt{-E_1})^2}{(k^2 - E_1^2)} = \frac{k + i\sqrt{-E_1}}{k - i\sqrt{-E_1}} \quad .$$

From (1.15) we see that the analytic continuation of f(k) is zero when $k = -i\sqrt{-E_1}$, as is required by the usual scattering theory.

As the next case, let us take $E_1 = 0$. We have, on solving (1.1) by separation of variables,

$$D(r,r') = C_1^{-1} \frac{|A_1|^2}{\pi} rr' , \tag{1.16}$$

$$K(r,r') = \frac{-3rr'}{r^3 + \lambda} , \tag{1.17}$$

where

$$\lambda = \frac{3C_1\pi}{|A_1|^2} , \tag{1.17a}$$

$$V = \frac{6r(r^3-2\lambda)}{(r^3+\lambda)^2} , \tag{1.18}$$

$$\Psi(r,k) = \sqrt{\frac{2}{\pi}} \left[\sin kr + \frac{3r^2}{r^3+\lambda} \frac{\cos kr}{k} - \frac{3r}{r^3+\lambda} \frac{\sin kr}{k^2} \right] , \tag{1.19}$$

and

$$\Psi(r,E_1) = \frac{A_1\lambda}{\sqrt{\pi}} \frac{r}{r^3 + \lambda} . \tag{1.20}$$

In the present case we see

$$\lim_{r\to\infty} \Psi(r,k) = \sqrt{\frac{2}{\pi}} \sin kr . \tag{1.21}$$

Hence the scattering phase is zero for all k, and we have accomplished our objective of obtaining such a potential. In the present case $f(k)$ is identically 1 on the whole complex k plane as is required.

It should be mentioned that the potential (1.18) was derived earlier by Bargmann [6] in his discussion of the inverse problem for the special case where the phase is a rational function of k.

Let us now consider the case for which $E_1 > 0$. Then

$$D(r,r') = \frac{C_1^{-1} |A_1|^2}{\pi \sqrt{E_1}} \sin \sqrt{E_1}\, r \sin \sqrt{E_1}\, r' . \tag{1.22}$$

$$(1.23) \qquad K(r,r') = \frac{-\sin\sqrt{E_1}\,r\,\sin\sqrt{E_1}\,r'}{\lambda + \frac{1}{2}\left(r - \frac{\sin 2\sqrt{E_1}\,r}{2\sqrt{E_1}}\right)} ,$$

where

$$(1.23a) \qquad \lambda = \frac{\pi C_1 \sqrt{E_1}}{|A_1|^2}$$

$$(1.24) \qquad V(r) = \frac{2\sin\sqrt{E_1}\,r}{\left[\lambda + \frac{1}{2}\left(r - \frac{\sin 2\sqrt{E_1}\,r}{2\sqrt{E_1}}\right)\right]^2}\left[\sin\sqrt{E_1}\,r - 2\sqrt{E_1}\left(\lambda + \frac{r}{2}\right)\cos\sqrt{E_1}\,r\right]$$

$$(1.25) \qquad \Psi(r,k) = \sqrt{\frac{2}{\pi}}\sin kr - \sqrt{\frac{2}{\pi}}\,\frac{\sin\sqrt{E_1}\,r\left(E_1 \sin kr\cos\sqrt{E_1}\,r - k\sin\sqrt{E_1}\,r\cos kr\right)}{\left[\lambda + \frac{1}{2}\left(r - \frac{\sin 2\sqrt{E_1}\,r}{2\sqrt{E_1}}\right)\right](k^2 - E_1)} .$$

$$(1.26) \qquad \Psi(r,E_1) = \frac{A_1\lambda}{\sqrt{\pi}\,(E_1)^{1/4}}\;\frac{\sin\sqrt{E_1}\,r}{\lambda + \frac{1}{2}\left(r - \frac{\sin 2\sqrt{E_1}\,r}{2\sqrt{E_1}}\right)} .$$

In the present case it is clear that

$$(1.27) \qquad \lim_{r\to\infty} \Psi(r,k) = \sqrt{\frac{2}{\pi}}\sin kr .$$

Hence again the scattering phase is zero. In the present case the potential behaves like $\frac{e^{\pm i\sqrt{E_1}\,r}}{r}$ for large r. Thus it is not a member of the class of potentials discussed in References [2] and [3] .

We can now discuss the case where there are several proper eigenstates present. In the manner similar to that above one can show generally that solutions of the Gelfand-Levitan equation for which $|f(k)| = 1$, and for which a finite number of non-negative point eigenvalues are assumed to exist lead to potentials with phaseless scattering. Conversely, of all potentials which support a finite number of bound states, these are the only non-trivial ones which lead to zero scattering phase for all k.

2. Higher angular momenta

The formulas for the radial equation corresponding to the general angular momentum ℓ are close to those for zero angular momentum.

The Gelfand-Levitan equation is

$$(2.1)\quad \left\{ \begin{aligned} K(r,r') &= -D(r,r') - \int_0^r K(r,r'')\, D(r'',r')\, dr'' \\ &\text{where} \\ D(r,r') &= \frac{2rr'}{\pi} \int_0^\infty \left[|f(k)|^{-2} - 1 \right] k^2 j_\ell(kr)\, j_\ell(kr')\,dk \\ &\quad + \sum_{i=1}^{n} C_i^{-1}\, g(r,E_i)\, g^*(r',E_i) \end{aligned} \right.$$

where the real numbers, E_i, correspond to the point eigenvalues. The functions $g(r,E_i)$ are given by

$$(2.1a)\quad \left\{ \begin{aligned} g(r,E_i) &= \frac{A_i}{\sqrt{\pi}} (-E_i)^{1/4} e^{i\pi/4} r j_\ell(i\sqrt{-E_i}\, r)\,, \quad (E_i < 0) \\ &= \frac{A_i}{\sqrt{\pi}} (E_i)^{1/4}\, r j_\ell(\sqrt{E_i}\, r), \quad (E_i > 0) \\ &= \frac{A_i}{\sqrt{\pi}}\, r^{\ell+1}, \quad (E_i = 0)\ . \end{aligned} \right.$$

As before, the constants C_i are the prescribed normalizations for the eigenstates corresponding to the point eigenvalues, while the constants A_i are arbitrary complex numbers which one must also prescribe.

Also, as before, one prescribes $|f(k)|$, the point eigenvalues E_i, and the normalization constants C_i and solves (2.1) for $K(r,r')$. One can find the potential $V(r)$, the eigenfunctions corresponding to the continuous spectrum $\Psi(r,k)$ and the eigenfunctions corresponding to the point spectrum $\Psi(r,E_i)$ from

$$
\begin{aligned}
V(r) &= 2\frac{d}{dr}K(r,r)\\
\Psi(r,k) &= \sqrt{\frac{2}{\pi}}\,kr\,j_\ell(kr) + \sqrt{\frac{2}{\pi}}\,k\int_0^r K(r,r')\,r' j_\ell(kr')dr' \qquad (2.2)\\
\Psi(r,E_i) &= g(r,E_i) + \int_0^r K(r,r')\,g(r',E_i)\,dr' \quad .
\end{aligned}
$$

The eigenfunctions satisfy the orthogonality and completeness relations (1.3) and (1.4) and are eigenfunctions of the operator $\frac{-d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r)$.

The scattering phase is related to $f(k)$ as in (1.5). The asymptotic form of $\Psi(r,k)$ is given by

$$
(2.3)\quad \lim_{r\to\infty}\Psi(r,k) = \sqrt{\frac{2}{\pi}}\,|f(k)|\,\sin(kr + \delta - \frac{\ell\pi}{2}) \quad .
$$

By the arguments of the previous section, zero scattering phase leads to a condition $f(k) \equiv 1$, but since only $|f(k)|$ is used in the Gelfand-Levitan equations, we must examine all solutions of the Gelfand-Levitan equation for which $|f(k)|\equiv 1$ to see which solutions lead to $f(k)\equiv 1$ or equivalently to zero phase potentials.

As before, we shall obtain phaseless potentials, if we assume a finite number of non-negative point eigenvalues. These will be the only potentials which are phaseless and which support a finite number of point eigenvalues.

The expressions for the potentials and eigenfunctions are calculated in a straightforward way, but they are somewhat cumbersome. Hence we shall write down only the simplest reflectionless potential and the corresponding eigenfunctions. This potential arises when one assumes a single point eigenvalue with the value zero and is analagous to the Bargmann potential of the previous section.

$$V(r) = \frac{2(2\ell+3)\, r^{2\ell+1}\left[r^{2\ell+3} - 2\lambda(\ell+1)\right]}{\left(r^{2\ell+3} + \lambda\right)^{2}}$$

$$\Psi(r,k) = \sqrt{\frac{2}{\pi}}\; rk\, j_{\ell}(rk) + \sqrt{\frac{2}{\pi}}\;\frac{1}{k}\;\frac{(2\ell+3)\, r^{2(\ell+1)}}{(r^{2\ell+3}+\lambda)}\left[kr j_{\ell}'(kr) - \ell j_{\ell}(kr)\right] \tag{2.4}$$

$$\Psi(r,E_1) = \frac{A_1 \lambda r^{\ell+1}}{\sqrt{\pi}(r^{2\ell+3}+\lambda)}$$

where

$$\lambda = \frac{(2\ell+3)\, C_1 \pi}{|A_1|^2} \quad . \tag{2.4a}$$

One might ask the question whether it is possible to choose a potential such that the phases for two or more angular momenta might not be identically zero. It is possible to show that in the class of potentials of the type considered in the present paper, namely for which f(k) is analytic in the lower half plane, this is not possible.

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